Precalculus Individual

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February 23 - March 9, 2019

The answer choice **(E)** NOTA denotes that "none of these answers" are correct. Tests are scored such that each correct answer is worth 4 points, each question left blank is worth 0 points, and each incorrect answer is -1 point. All answers must be exact unless otherwise specified. All angles are in radians unless otherwise stated. For all matrices M with nonzero determinant, $M^0 = I$. DNE denotes that a function, value, or answer "does not exist", and $i = \sqrt{-1}$. Good luck, and more importantly, have fun!

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1. Let α represent the measure of an angle, such that $\tan\alpha = \frac{4}{5}$. Evaluate $\sin\alpha\cos\alpha$.

(A) $\frac{20}{41}$ **(B)** $\frac{12}{25}$ **(C)** $\frac{16}{25}$ **(D)** $\frac{20}{81}$ **(E)** NOTA

2. A particular obtuse triangle $\triangle ABC$ has $AB = 10$ units and $BC = 24$ units. Given that $\angle B = \arccos\left(-\frac{24}{25}\right)$, what is the area of triangle $\triangle ABC$, in square units?

(A) 33.6 **(B)** 48 **(C)** 67.2 **(D)** 96 **(E)** NOTA

3. Find the sum of the coefficients of the standard form of the equation of an ellipse centered about the point $(1, -3)$ with a horizontal axis of length 8 and vertical axis of length 6.

(A) 96 **(B)** 112 **(C)** 128 **(D)** 132 **(E)** NOTA

4. Find the cotangent of the angle between the two vectors $\overrightarrow{u} = \langle 1, -2, 3\rangle$ and $\overrightarrow{v} = \langle -3, -1, 2\rangle$.

(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{2\sqrt{3}}{3}$ **(C)** $\frac{5\sqrt{19}}{57}$ **(D)** $\frac{3\sqrt{19}}{5}$ **(E)** NOTA

5. Find the sum of the solutions, where $x \in [0, 2\pi]$, of the equation $\sin^2(2x) \cdot \tan(x) \cdot \cot(2x) = 0$.

(A) 0 **(B)** 2π **(C)** 7π **(D)** 16π **(E)** NOTA

6. The simplest form of $(1+i)^{69}$ can be expressed as $-a^b(1+i)$, where a is a prime number and b is a positive integer. What is ab?

(A) 36 **(B)** 70 **(C)** 136 **(D)** 138 **(E)** NOTA

7. Find the area for the curve defined by the graph of the parametric equations $x(t) = 5\cos(t) + 7$ and $y(t) = 4\sin(t) + 2$ on the Cartesian plane.

(A) 10π **(B)** 20π **(C)** 80π **(D)** 400π **(E)** NOTA

8. If $\cos(x) = \frac{\sqrt{5}}{5}$, what is $\tan(4x)$?

(A) $-\frac{45}{11}$ **(B)** $\frac{24}{7}$ **(C)** $\frac{45}{11}$ **(D)** 8 **(E)** NOTA

9. $1 - 1 + 1 - 1 + 1 - 1 + \ldots =?$

(A) 0 **(B)** 0.5 **(C)** 1 **(D)** DNE **(E)** NOTA

10. What is $\left\lceil \sum_{i=1}^{2019} \ln\left(\frac{203}{202}\right) \right\rceil$?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** NOTA

11. Given the quartic sequence $1, 3, 8, 17, 32, \ldots$, what is the next term?

(A) 56 **(B)** 49 **(C)** 41 **(D)** 37 **(E)** NOTA

12. How many solutions does $\sin(x) + \cos(x) + \csc(x) + \sec(x) + \tan(x) + \cot(x) = 2\csc(2x)$ have over the domain $[-2\pi, 2\pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** NOTA

Question 13 refers to the following functions.

$$f(x) = e^{(x^2)} - 1$$
$$g(x) = \ln\left(\sqrt{x^2 - 1}\right) - \ln\left(\sqrt{x - 1}\right)$$
$$h(x) = \sqrt{2x}$$

13. What is $\ln(h(g(f(e))))$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** e **(E)** NOTA

14. Let a conic section's "cool" factor be defined as:

2: If it's a parabola
3: If it's a circle
5: If it's a non-circle ellipse
7: If it's a hyperbola
1: If it's degenerate

Given $x^2 + 4xy + 3y^2 + 2x + 6y - 18 = 0$, what is the product of this graph's cool factor and the smallest angle (in degrees) it can be rotated to have a horizontal axis?

(A) 270 **(B)** 315 **(C)** 450 **(D)** 630 **(E)** NOTA

15. How many times does $y = 2\sin(x)$ intersect $y = \ln|x|$?

(A) 31 **(B)** 32 **(C)** 63 **(D)** 64 **(E)** NOTA

16. What is the magnitude of the complex solution to $x^{12} - 2019 = 0$ with the largest real part, and a positive imaginary coefficient?

(A) 0 **(B)** 1 **(C)** 2019 **(D)** 2020 **(E)** NOTA

17. The perfectly spherical earth has a volume of 1 trillion km^3, then Shaggy squished the earth with 0.1% of his power, to fit in a sandwich, where the new vertical diameter is now $\frac{150}{\pi}$ km. If he managed to maintain the initial volume, and squished it evenly into an ellipsoid, what would be the farthest Euclidean distance between two people on earth?

(A) 50,000 **(B)** 100,000 **(C)** 200,000 **(D)** 400,000 **(E)** NOTA

18. How many of the following statements are true about $f(x) = \dfrac{x \cdot |x|}{x^2 + 1}$?

i. $f(x)$ is continuous
ii. $f(x)$ is invertible about all reals
iii. the domain of the inverse of $f(x)$ is finite
iv. $f(x)$ is smooth over all reals (no sharp turns)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NOTA

19. What mathematical construct is most commonly associated with the graph of a leminiscate?

(A) Zero **(B)** Golden Ratio **(C)** Pi
(D) Infinity **(E)** NOTA

Question 20 and 21 refer to the following information.

Jake and Logan have an infinite plane that has no weight and cannot be manipulated. Jake throws a t-shirt on the plane that weighs 5 lbs, at $(4, 4)$, then Logan throws a hoodie at the point $(3, -6)$ that weighs 7 lbs, Jake wants to show off, so then he throws the keys to his Lambo that weigh 3 lbs at $(0, 8)$. Logan wants to prove to his brother that he is smarter because he has an engineering degree, so he will balance the whole plane on a stick.

20. What is the tangent of the angle that is formed with the x-axis and the line that connects the origin of the plane to the point where Logan perfectly balances the plane?

(A) $\frac{1}{21}$ **(B)** $\frac{1}{14}$ **(C)** $\frac{1}{7}$ **(D)** $\frac{1}{2}$ **(E)** NOTA

21. The planar distance from where Logan would have to position the stick, to the origin of the plane can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b has no perfect square factors besides 1. What is $a + b + c$?

(A) 216 **(B)** 238 **(C)** 254 **(D)** 276 **(E)** NOTA

22. Let A be a 2019 x 2019 identity matrix, and B be a vector matrix with 2019 rows, where each row argument is its row number. Constant $l =$ the determinant of A, $i =$ the sum of the elements of AB, and $t =$ the reciprocal of the trace of A. What is lit?

(A) 2020 **(B)** 2019 **(C)** 1010 **(D)** 1009 **(E)** NOTA

23. If F_n is the nth Fibonacci number, what is $\lim_{n\to\infty} \ln(F_{n+1}) - \ln(F_n) - \dfrac{\ln(6+2\sqrt{5})}{2}$?

(A) $-\ln 2$ **(B)** $\ln 2$ **(C)** 0 **(D)** 1 **(E)** NOTA

24. A sphere of radius 6 is centered about the origin, what is the largest value of the variable "*yeet*," given the equation $xz = yeet$ intersects the sphere at least once?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** NOTA

25. What is $\text{cis}(-1)^{\text{cis}(\frac{\pi}{2})}$?

(A) 0 **(B)** 1 **(C)** -1 **(D)** i **(E)** NOTA

26. Ninja just found a legendary rocket launcher, and wants to use it on some of his unsuspecting "stream snipers" before the delay catches up. He holds the launcher at a height of 4 feet and a 30 degree angle upwards, the initial velocity of the rocket is 12 feet per second. If he wants to head-shot someone on a platform 2 feet below him, who is 5 feet tall, what should be his horizontal distance be, in feet?

(A) 0.25 **(B)** 0.375 **(C)** 0.5 **(D)** 0.625 **(E)** NOTA

27. The Pokemon Machamp has one of his four hands, that he named "Bessy," at the origin, holding the corner of a parallelepiped, his other three hands are holding the opposite corners of the three faces of the box that Bessy is touching. His other hands are at the points $(1, 2, 3)$, $(3, 2, 1)$ and $(4, 4, 4)$, what is the volume of the box, in cubic units?

(A) 8 **(B)** $12\sqrt{2}$ **(C)** 16 **(D)** $16\sqrt{6}$ **(E)** NOTA

28. Evaluate the limit

$$\lim_{h\to 0} \frac{(x+7h)^8 - (x-2h)^8}{3h}.$$

(A) 6 **(B)** 12 **(C)** 24 **(D)** 48 **(E)** NOTA

29. In calculus, you will learn about p-series for testing convergence of harmonic series, but right now, let's look at t-series. A t-series is defined as $\left(\frac{1}{2}\right)^t - \left(\frac{1}{4}\right)^t + \left(\frac{1}{8}\right)^t - \left(\frac{1}{16}\right)^t \ldots$ For what domain of t is the t-series between $\frac{1}{3}$ and $\frac{2}{3}$?

(A) $(-\infty, -1) \cup (1, \infty)$ **(B)** $(-\infty, -1] \cup [1, \infty)$ **(C)** $[-1, 1)$
(D) $(-1, 1)$ **(E)** NOTA

30. Let a represent the amplitude of the sinusoidal function $f(x) = -2\cos(4x + 7)$. Let b represent the period of the same function. Find ab.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 8 **(D)** 16 **(E)** NOTA