2019 AoPS Mock MAΘ Convention

July 1-13, 2019

AoPS Convention

2019

Theta Individual

This is a 30 question, 60 minute long test. Scoring is 5 times the number of correct answers plus the number of questions left blank. An incorrect answer is worth 0 points. Read all directions and questions carefully.

The problems on this test were created by:

Archeon
reddragon644
Math1331Math
jybsmartguy2

The answer choice ***(E)*** *NOTA denotes that "none of these answers" are correct. All answers must be exact unless otherwise specified. Assume all functions are real-valued unless otherwise specified. When answers are given in a certain form, put them in the most intuitive way to write that form; please do not attempt to dispute alternate forms. Good luck, and have fun!*

1. An insurance company promises that you can save "up to 15% or more" on your insurance. Which of the following intervals includes all possible percentages that would be consistent with the promise?

(A) $[0\%, 15\%]$ **(B)** $[0\%, 15\%)$ **(C)** $[15\%, 100\%]$ **(D)** $[0\%, 100\%]$ **(E)** NOTA

2. If $f(x) = x + 1$ and $f^n(x)$ denotes n compositions of f (e.g. $f^2(x) = f(f(x))$), what is $f^{2019}(1)$?

(A) 2018 **(B)** 2019 **(C)** 2020 **(D)** 2021 **(E)** NOTA

3. Let $ABCD$ be a square with side length 4. Let E and F be the midpoints of $\overline{AB}$ and $\overline{CD}$ respectively. What is the area of quadrilateral $AECF$?

(A) 4 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** NOTA

4. A *Mersenne prime* is a prime number that can be written in the form $2^n - 1$, where n is a positive whole number. Find the sum of the digits of the smallest three-digit Mersenne prime.

(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** NOTA

5. Carolina is selling popsicles of two different sizes. She sells small popsicles for $1 each and large popsicles for $2 each. If she sells 100 popsicles and makes $135, how many large popsicles did she sell?

(A) 35 **(B)** 45 **(C)** 55 **(D)** 65 **(E)** NOTA

6. Connor and Jae often play League of Legends together. Whenever they play together, they win 80% of their games. In contrast, when Jae plays without Connor, he only wins 50% of his games. If Jae plays $\frac{2}{3}$ of his ranked games with Connor, what is the percentage of all of his games that Jae wins?

(A) 60% **(B)** 65% **(C)** 70% **(D)** 75% **(E)** NOTA

7. Odin runs past Nico while travelling in a straight line at a constant speed of 3 m/s. 10 seconds after Odin passes him, Nico chases after Odin at a constant speed of 5 m/s. For how many seconds has Nico been running when he catches up to Odin?

(A) 6 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** NOTA

8. A parabola has a vertex at $(20, 19)$ and an x intercept at $(0, 0)$. If the parabola is written in the form $y = ax^2 + bx + c$, then a can be expressed in simplest form as $-\frac{p}{q}$. What is $p + q$?

(A) 17 **(B)** 63 **(C)** 107 **(D)** 419 **(E)** NOTA

9. Chanakya has a cube with length six centimeters that he cherishes a lot. Ilias is jealous and ends up breaking this cube into twenty-seven smaller cubical pieces, with each new piece being two centimeters long. Find the ratio of the surface area of the original cube to the total surface area of all the individual pieces.

(A) $\frac{1}{6}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** NOTA

10. Alex places his toy boat in a river that flows at a constant speed. After it drifts for one minute, the boat turns on its propellor and begins to head back towards Alex. 40 seconds later, it reaches Alex at the point it was initially placed. Find the ratio of the speed of the boat in still water to the speed of the river.

(A) $\frac{3}{2}$ **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** NOTA

11. Find the sum of the coordinates of the focus of the conic given by $16x^2 + 25y^2 - 32x - 100y - 284 = 0$ that has a larger x-coordinate.

(A) 3 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NOTA

12. Consider two equilateral triangles $\mathcal{T}$ and $\mathcal{T}'$ such that $\mathcal{T}'$ is the result of rotating $\mathcal{T}$ by $60°$ about its center. Find the ratio of the area of the union of $\mathcal{T}$ and $\mathcal{T}'$ to the area of $\mathcal{T}$ (The *union* of two figures $\mathcal{A}$ and $\mathcal{B}$ is the set of all points inside of $\mathcal{A}$, $\mathcal{B}$, or both).

(A) $\frac{4}{3}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{3}$ **(D)** 2 **(E)** NOTA

13. Find the number of solutions to $|x - 3|x| + 10| = 6$.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NOTA

14. The area of the first quadrant region satisfying both $x \le \sqrt{36 - y^2}$ and $x \le 3$ can be expressed as $a\sqrt{3} + b\pi$ for rational numbers a and b. What is $a + b$?

(A) 3.75 **(B)** 5.25 **(C)** 6.5 **(D)** 7.5 **(E)** NOTA

15. The sum of the lengths of the two legs of a right triangle is $2\sqrt{5}$. The area of this triangle is 1. Find the length of the hypotenuse of this right triangle.

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $2\sqrt{5}$ **(D)** $2\sqrt{6}$ **(E)** NOTA

16. There exists one real number x satisfying $(\log_5 x)^3 - (\log_5(x^{\sqrt{3}}))^2 + \log_5(x^3) = 1$. What is x?

(A) $\sqrt{5}$ **(B)** 5 **(C)** $5\sqrt{5}$ **(D)** 25 **(E)** NOTA

17. Chords $\overline{AC}$ and $\overline{BD}$ of circle Ω intersect at X such that $\overline{AC} \perp \overline{BD}$. If $XA = XC = 6$ and $XB = 2$, what is the radius of circle Ω?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** NOTA

18. For real-valued x, the maximum real value of $\sqrt{10+x}+\sqrt{42-x}$ can be expressed as $\sqrt{n}$ for some positive integer n. Find the sum of the digits of n.

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** NOTA

19. Arnav is creating a head-to-head math tournament for his Theta students. If he plans on having 16 competitors in a single elimination bracket (that is, a student is eliminated after losing one round), and each round is a "best of 3" problem round (that is, the first person to get 2 problems wins the round), what is the minimum number of problems Arnav needs to write so he can always determine a single champion? Assume that exactly one student solves each problem.

(A) 30 **(B)** 32 **(C)** 45 **(D)** 48 **(E)** NOTA

20. If A and B are real numbers such that

$$\frac{8x-7}{2x^2-5x+2}=\frac{A}{x-2}+\frac{B}{2x-1},$$

what is $2A+3B$?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** NOTA

21. Vlad has two distinct lines ℓ and m that intersect at a point. He reflects ℓ over m, resulting in a new line ℓ'. He then reflects m over ℓ', resulting in another new line m'. He continues this process of reflecting one line over the other in an alternating fashion until he realizes that he has looped around and obtained lines ℓ and m again. Given that he made a total of 12 reflections to reach this point, what is the smallest possible acute angle between the two lines?

(A) 15° **(B)** 20° **(C)** 30° **(D)** 45° **(E)** NOTA

22. Let matrix $A_{a,b}$ be defined as

$$A_{a,b}=\begin{bmatrix} 5 & b & 4 \\ 12 & 8 & 4 \\ a & 6 & 3 \end{bmatrix}$$

where a and b are real numbers. There exists a k such that all matrices $A_{k,b}$ have no inverse. What is k?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** NOTA

23. Let $\{a_n\}$ be a geometric series with first term a_1. If

$$\sum_{n=1}^{\infty} a_{2n-1}=4 \quad \text{and} \quad \sum_{n=1}^{\infty} a_{2n}=3,$$

what is a_1?

(A) $\frac{3}{4}$ **(B)** 1 **(C)** $\frac{7}{4}$ **(D)** $\frac{21}{4}$ **(E)** NOTA

24. Rectangle $ABCD$ has $AB = 6$ and $BC = 4$. Points E and F lie on sides $\overline{BC}$ and $\overline{CD}$, respectively, such that $BE = 1$ and $CF = 4$. Line ℓ passes through A and E, while line m passes through points B and F and intersects line ℓ at point X. Find the area of $\triangle BXE$.

(A) $\frac{2}{7}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{4}{7}$ **(D)** $\frac{6}{7}$ **(E)** NOTA

25. A cubic function f satisfies $f(x) = x$ for $x = 1, 2, 3$. In addition, $f(4) = 10$. What is the units digit of $f(2019)$?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** NOTA

26. Quadrilateral $ABCD$ has side lengths $AB = 13$, $BC = 12$, $CD = 8$, and $AD = 7$. If one of the diagonals has length 5, the area of quadrilateral $ABCD$ can be expressed in simplest form as $a + b\sqrt{c}$. What is $a + b + c$?

(A) 43 **(B)** 73 **(C)** 103 **(D)** 133 **(E)** NOTA

27. Two circles of equal radius with centers A and B are externally tangent to each other and internally tangent to a larger circle with center O. If $\angle AOB = 60^\circ$, what fraction of the area of the larger circle is not within either of the two smaller circles?

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{5}{7}$ **(D)** $\frac{7}{9}$ **(E)** NOTA

28. Triangle ABC has a right angle at A. Let E and F lie on hypotenuse $\overline{BC}$ such that $BE = EF = FC$ and E is closer to B than C. If $AE = 20$ and $AF = 25$, then BC can be expressed in simplest form as $m\sqrt{n}$. What is $m + n$?

(A) 46 **(B)** 50 **(C)** 208 **(D)** 210 **(E)** NOTA

29. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. The graph of the function $y = \{x \cdot \lfloor x \rfloor\}$ from $x = 0$ to $x = 10$ consists of n distinct line segments[1] . What is n?

(A) 45 **(B)** 46 **(C)** 55 **(D)** 56 **(E)** NOTA

30. What is the minimum value of

$$\frac{a + b + c}{a^2 + b^2 + c^2 + 3}$$

where a, b, and c are real numbers?

(A) -1 **(B)** $-\frac{3}{4}$ **(C)** $-\frac{2}{3}$ **(D)** $-\frac{1}{2}$ **(E)** NOTA

[1] For the pedantic among you, do not attempt to dispute the question by saying you can split a line segment into two smaller subsegments, therefore making n not uniquely defined. It will not be accepted. Assume "line segment" refers to an entire line segment.