2019 AoPS Mock MAΘ Convention

July 1-13, 2019

AoPS Convention

2019

Mu Individual

This is a 30 question, 60 minute long test. Scoring is 5 times the number of correct answers plus the number of questions left blank. An incorrect answer is worth 0 points. Read all directions and questions carefully.

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*The answer choice **(E)** NOTA denotes that "none of these answers" are correct. All answers must be exact unless otherwise specified. Assume all functions are real-valued unless otherwise specified. DNE stands for "does not exist". Have fun!*

1. Evaluate the following limit.

$$\lim_{x\to 3}\left(\frac{x^2-4x+3}{x^2-5x+6}\right)$$

(A) $\frac{1}{2}$ **(B)** $\frac{4}{5}$ **(C)** 2 **(D)** DNE **(E)** NOTA

2. Find the slope of the tangent line to the graph of $y = 20x^{19}$ at $x = 1$.

(A) 19 **(B)** 20 **(C)** 39 **(D)** 380 **(E)** NOTA

3. Find the sum of the digits of the maximum value of $2x^3 - 15x^2 + 24x + 100$ for $0 \le x \le 5$.

(A) 1 **(B)** 3 **(C)** 12 **(D)** 14 **(E)** NOTA

4. Find the area bounded by the first-quadrant portion of $y = -x^2 + 8x - 7$.

(A) 24 **(B)** 36 **(C)** 48 **(D)** 54 **(E)** NOTA

5. If the volume of the solid formed when the finite region bounded by the graph of $y = x^2 - 5x + 6$ and the x-axis is rotated about the y-axis can be expressed in simplest form as $\frac{m\pi}{n}$, what is $m + n$?

(A) 11 **(B)** 13 **(C)** 17 **(D)** 31 **(E)** NOTA

6. Saaketh's swimming pool has a very unique shape. It is the shape of a conical frustum with upper radius 30 feet and lower radius 10 feet. The height of the pool is 100 feet (indeed many would claim Saaketh's pool to have the depth of a lake)! Periodically, the pool needs to be drained for cleaning. At the bottom of the pool, Saaketh has a drain that when opened drains the water at 360π cubic feet per second. When the height of the water in the pool is 50 feet, at what rate is the height of the pool decreasing, in feet per second?

(A) $\frac{4}{5}$ **(B)** $\frac{9}{10}$ **(C)** $\frac{8}{5}$ **(D)** $\frac{9}{5}$ **(E)** NOTA

7. Evaluate the following limit.

$$\lim_{x\to\infty}\left(x\sqrt{x^2+x+5} - x\sqrt{x^2+x}\right)$$

(A) 0 **(B)** $\frac{5}{2}$ **(C)** 5 **(D)** DNE **(E)** NOTA

8. Evaluate the following definite integral.

$$\int_0^1 (1-x)^{10}x^2\,dx$$

(A) $\frac{1}{858}$ **(B)** $\frac{1}{364}$ **(C)** $\frac{1}{110}$ **(D)** $\frac{1}{66}$ **(E)** NOTA

9. A sequence a_n is given by $a_0 = 2$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geq 0$. As n grows large, a_n approaches a constant value k. What is k?

(A) 1 **(B)** $\frac{1-\sqrt{5}}{2}$ **(C)** $\frac{1+\sqrt{5}}{2}$ **(D)** $\frac{3+\sqrt{5}}{2}$ **(E)** NOTA

10. A particle P moves along the graph of $y = x^2$ such that the x-coordinate $x(t)$ of P at time t is $2t$. Let $\ell(t)$ denote the x-intercept of the tangent line to $y = x^2$ at P at time t. Evaluate $\ell'(2)$.

(A) $\frac{1}{2}$ **(B)** 1 **(C)** 2 **(D)** 4 **(E)** NOTA

11. A rectangular prism-shaped box with a square base and an open top has a surface area of 288. What is the maximum possible volume of the box?

(A) $96\sqrt{6}$ **(B)** $192\sqrt{6}$ **(C)** $384\sqrt{6}$ **(D)** $768\sqrt{6}$ **(E)** NOTA

12. Let $\{x\}$ denote the fractional part of x. Evaluate

$$\int_0^{120} \{x\}\{x + \tfrac{1}{2}\}\, dx.$$

(A) 25 **(B)** 30 **(C)** 35 **(D)** 40 **(E)** NOTA

13. The area of the region bounded by the graphs of $r = \cos(\theta) + \sin(\theta)$ and $r = 1$ can be expressed as $a + b\pi$, where a and b are rational numbers. What is $|a| + |b|$?

(A) 1 **(B)** $\frac{5}{4}$ **(C)** $\frac{3}{2}$ **(D)** 2 **(E)** NOTA

14. A point P is selected uniformly at random inside the square with opposite vertices at $(0,0)$ and $(1,1)$. What is the probability that this point is closer to $(0,0)$ than it is to the line $x = 1$?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** NOTA

15. Evaluate the following definite integral.

$$\int_0^1 \frac{\arctan(x)}{x^2+1}\, dx$$

(A) $\frac{\pi^2}{32}$ **(B)** $\frac{\pi^2}{16}$ **(C)** $\frac{\pi^2}{8}$ **(D)** $\frac{\pi^2}{4}$ **(E)** NOTA

16. A curve is defined parametrically by

$$x(t) = \cos(t), \qquad y(t) = \sin^2(t).$$

If the area bound by this curve and the x-axis can be expressed in simplest form as $\frac{m}{n}$, what is $m + n$?

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** NOTA

17. How many of the following series converge?

$$\text{i. } \sum_{n=2}^{\infty} \frac{1}{n\ln(n)\ln(\ln(n))} \qquad \text{ii. } \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \text{iii. } \sum_{n=1}^{\infty} \left(\frac{n^2+2n+1}{2n^2+n+2}\right)^n \qquad \text{iv. } \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\ln(n)}$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NOTA

18. Find the largest possible area of a trapezoid that can be inscribed in a semi-circle of radius 4.

(A) 16 **(B)** $12\sqrt{3}$ **(C)** $8+4\sqrt{3}$ **(D)** $8+8\sqrt{2}$ **(E)** NOTA

19. Let $f(x)$ be a quartic function such that $f(2) = f'(2) = f''(2) = 2$. If, $R(x)$ is the remainder when $f(x)$ is divided by $(x-2)^3$, evaluate $R(3)$.

(A) 2 **(B)** 5 **(C)** 11 **(D)** 17 **(E)** NOTA

20. There exists a line of positive slope m that is tangent to the graphs of both $y = x^2+2x+2$ and $y = -x^2+2x-2$. Find the greatest integer less than or equal to m.

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** NOTA

21. Let $s_1(x) = \sin(x)$ and for $n > 1$ let $s_n(x) = \sin(s_{n-1}(x))$. Evaluate

$$\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} s_n(x)\,dx.$$

(A) 0 **(B)** $\frac{1}{\pi}$ **(C)** $\frac{2}{\pi}$ **(D)** $\frac{\pi}{4}$ **(E)** NOTA

22. Let the 8th derivative of $\ln(x+\sqrt{x^2+1})$ at $x=0$ be K. What is the greatest integer less than or equal to $100K$?

(A) 57 **(B)** 100 **(C)** 141 **(D)** 173 **(E)** NOTA

23. Consider the relationship $y^n = x^n + xy$. As n grows large, what is the value of $\frac{dy}{dx}$ at $x = 2019$, $y > 0$?

(A) -1 **(B)** 1 **(C)** $\ln(2019)$ **(D)** 2019 **(E)** NOTA

24. Let L be defined by the following limit.

$$L = \lim_{x\to 0} \frac{1-(\cos(x)\cos(2x)\cos(3x)\cdots\cos(40x))}{x^2}$$

What is the remainder when L is divided by 100?

(A) 40 **(B)** 50 **(C)** 60 **(D)** 70 **(E)** NOTA

25. Find the real constant α such that

$$\int_0^\infty \frac{\cos^{2019} x + \alpha}{x^2}\,dx$$

is finite.

(A) -2019 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** NOTA

26. A function $y(x)$ satisfies the differential equation

$$2xy = x^3y^2 + x^2y' + 5y^2.$$

If $y(2) = 1$, what is the greatest integer less than or equal to $5y(1)$?

(A) -8 **(B)** -7 **(C)** -6 **(D)** -5 **(E)** NOTA

27. Evaluate the following definite integral.

$$\int_1^\infty \frac{\lfloor \sin(x) + 1 \rfloor}{x^2}\,dx$$

(A) $1 - \frac{\ln(2)}{\pi}$ **(B)** $1 - \frac{1}{\pi}$ **(C)** $1 + \frac{\ln(2)}{\pi}$ **(D)** $1 + \frac{1}{\pi}$ **(E)** NOTA

28. Evaluate the following sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n(2n-1)}$$

(A) $\frac{\pi\sqrt{3}}{18}$ **(B)** $\frac{\pi}{6}$ **(C)** $\frac{\pi\sqrt{3}}{9}$ **(D)** $\frac{\pi}{3}$ **(E)** NOTA

29. Evaluate the following definite integral, given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

$$\int_{-\infty}^{\infty} \frac{2019^{-(x^2-x)}}{2019^x + 1}\,dx$$

(A) $\frac{1}{2}\sqrt{\frac{\pi}{\ln(2019)}}$ **(B)** $\sqrt{\frac{\pi}{\ln(2019)}}$ **(C)** $\frac{\sqrt{\pi}}{2\ln(2019)}$ **(D)** $\frac{\sqrt{\pi}}{\ln(2019)}$ **(E)** NOTA

30. Find the real number α that satisfies the equation

$$\lim_{n\to\infty} \alpha\sqrt{n} \int_0^{\frac{\pi}{2}} (\sin x)^{2n}\,dx = 1.$$

(A) $\sqrt{2}$ **(B)** $\sqrt{2\pi}$ **(C)** $2\sqrt{\pi}$ **(D)** $2\sqrt{2\pi}$ **(E)** NOTA