2019 AoPS Mock MAΘ Convention

July 1-13, 2019

AoPS Convention

2019

Alpha Individual

This is a 30 question, 60 minute long test. Scoring is 5 times the number of correct answers plus the number of questions left blank. An incorrect answer is worth 0 points. Read all directions and questions carefully.

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*The answer choice **(E)** NOTA denotes that "none of these answers" are correct. All answers must be exact unless otherwise specified. Assume all functions are real-valued unless otherwise specified. Diagrams are not necessarily to scale. Have fun!*

1. Find the smallest positive value of x satisfying $\sin^2(x) = \cos^2(x)$ but not $\sin(x) = \cos(x)$.

(A) $\frac{\pi}{4}$ **(B)** $\frac{3\pi}{4}$ **(C)** $\frac{5\pi}{4}$ **(D)** $\frac{7\pi}{4}$ **(E)** NOTA

2. Find the area enclosed by the graph of $r = 2\cos(\theta)$.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 2π **(D)** 4π **(E)** NOTA

3. Evaluate $|4i + |3i + |2i + 1|||^2$.

(A) 30 **(B)** 41 **(C)** 82 **(D)** 100 **(E)** NOTA

4. Vlad is spinning in a circle at 1 revolution per second while holding his whacking stick. Suddenly, Odin pops into existence at some point selected uniformly at random within the circular region in reach of Vlad's whacking stick. What is the expected amount of time it takes Vlad to whack Odin?

(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** NOTA

5. The entries of a 2×2 matrix are -1, 0, or 1. What is the maximum possible determinant of this matrix?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** NOTA

6. Find the sum of all values of $\theta \in [0, 2\pi]$ satisfying $\sin(\theta) + \cos^2(\theta) = \cos(\theta) + \sin^2(\theta)$.

(A) 2π **(B)** 3π **(C)** 4π **(D)** 5π **(E)** NOTA

7. A linear function f with real coefficients satisfies $f(f(f(x))) = 8x + 21$. Compute $f(1)$.

(A) 5 **(B)** 7 **(C)** 9 **(D)** 11 **(E)** NOTA

8. If $\sin(\theta) + \cos(\theta) = \frac{\sqrt{5}}{2}$, what is $\cot^2(2\theta)$?

(A) 3 **(B)** 5 **(C)** 11 **(D)** 15 **(E)** NOTA

9. A finite curve is defined by the parametric equations $x(t) = \sin(t)$ and $y(t) = \cos^2(t)$. Find the smallest k such that $0 \le t \le k$ will trace the entire curve.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** $\frac{3\pi}{2}$ **(D)** 2π **(E)** NOTA

10. The function
$$y = \frac{ax+b}{x^2+cx+d}$$
has a vertical asymptote at $x = 3$ and a removable discontinuity at $(2, -4)$. What is $a+b+c+d$?

(A) 5 **(B)** 9 **(C)** 23 **(D)** 27 **(E)** NOTA

11. Let α and β be second quadrant angles satisfying $\sin(\alpha) = \frac{3}{5}$ and $\tan(\beta) = -1$. Compute $\sin(\alpha + \beta)$.

(A) $-\frac{7\sqrt{2}}{10}$ **(B)** $-\frac{\sqrt{2}}{10}$ **(C)** $\frac{\sqrt{2}}{10}$ **(D)** $\frac{7\sqrt{2}}{10}$ **(E)** NOTA

12. If a and b are positive real numbers such that $a + b = 24$, what is the maximum possible area of the ellipse given by $ax^2 + by^2 = ab$?

(A) $2\pi\sqrt{3}$ **(B)** $2\pi\sqrt{6}$ **(C)** 12π **(D)** 24π **(E)** NOTA

13. Two boats start at the same point. At 12:00 pm, one boat heads out at 2 mph due east. At 1:30 pm, the other boat heads out at 3 mph at an angle of 30° west of north. At 2:30 pm, how far apart are the boats, in miles?

(A) $\sqrt{19}$ **(B)** $\sqrt{34}$ **(C)** 7 **(D)** 8 **(E)** NOTA

14. The polynomial $x^3 - 2x^2 - 3x - 1$ has one real root r. Find $\lfloor r \rfloor$. (The greatest integer function $\lfloor x \rfloor$ returns the greatest integer less than or equal to x).

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** NOTA

15. Compute the period of the function $\cos(4\cos(x) + 3\sin(x))$.

(A) $\frac{\pi}{5}$ **(B)** $\frac{2\pi}{5}$ **(C)** π **(D)** 2π **(E)** NOTA

16. From a point (x, y), Kev the frog can jump to either $(x+2, y+1)$ or $(x+3, y+1)$. How many distinct paths can Kev take to get from $(0, 0)$ to $(17, 7)$?

(A) 21 **(B)** 35 **(C)** 56 **(D)** 120 **(E)** NOTA

17. Vincent has a drawer with two identical-looking digital clocks. One is a 12 hour clock (a normal clock) and the other is a 24 hour clock (displays, for example, 13:00 at 1:00 pm). He picks one from the drawer and puts it on his desk. When he turns it on, he sees that it reads 2:37 (he has been inside all day, so he has no other information as to what time it is). What is the probability that he selected the 12 hour clock?

(A) $\frac{1}{3}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** NOTA

18. A point (x, y) is rotated by an angle of $90°$ counter-clockwise about the origin and then translated 4 units right and 3 units down. If the resulting point is also (x, y), what is $x + y$?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 7 **(E)** NOTA

19. In the complex plane, the roots of

$$z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0,$$

when connected in clockwise order, form a polygon. What is the area of this polygon?

(A) $\frac{1+3\sqrt{2}}{2}$ **(B)** $2\sqrt{2}$ **(C)** $1 + 3\sqrt{2}$ **(D)** $4\sqrt{2}$ **(E)** NOTA

20. How many 2×2 matrices M satisfy the equation $2M - M^2 = I^5$, where I is the 2×2 identity matrix?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** NOTA

21. How many distinct intersection points are there between the graphs of $r = \theta$ and $25x^2 + y^2 = 100$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** NOTA

22. The ellipse given by $x^2 + 9y^2 = 9$ is rotated by an angle of $\frac{\pi}{2}$ counter-clockwise. This new ellipse intersects the original ellipse at four points. What is the area of the convex quadrilateral formed by connecting these four points?

(A) $\frac{9}{10}$ **(B)** $\frac{3}{2}$ **(C)** $\frac{9}{5}$ **(D)** 3 **(E)** NOTA

23. A sequence of matrices M_n is given by

$$M_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_{n+1} = M_0 M_n + \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} \quad \text{for all } n \geq 0.$$

If the sum of the elements of M_{2019} is S, compute $\lceil \log_2(S) \rceil$ (where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x).

(A) 2019 **(B)** 2020 **(C)** 2021 **(D)** 2022 **(E)** NOTA

24. Consider the graphs of the polar equations $r = 1 + \cos(\theta)$ and $r = a\cos(2\theta)$. For $a > 2$, these graphs intersect at nine points, one of which is the origin. The eight non-origin intersection points are connected in clockwise order to form a convex octagon. As a grows to infinity, the area of this octagon approaches k. What is k?

(A) 1 **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{5}{2}$ **(E)** NOTA

25. If $\frac{\sin(3x)}{\sin(x)} = \frac{5}{2}$, then $\frac{\cos(3x)}{\cos(x)}$ can be expressed in simplest form as $\frac{m}{n}$. What is $m+n$?

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** NOTA

26. A curve $\mathcal{C}$ is defined parametrically by the equations

$$x(t) = t^3 - 4t^2 - t + 4, \quad y(t) = t^2 - 4t.$$

$\mathcal{C}$ intersects itself one time at a point (a, b). What is $a+b$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NOTA

27. The *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S. Define a set $K = \{1, 2, \text{oatmeal}\}$. How many elements of $\mathcal{P}(\mathcal{P}(K))$ contain at least one set with at least one numerical element (assume that oatmeal is not a number)?

(A) 704 **(B)** 712 **(C)** 716 **(D)** 718 **(E)** NOTA

28. A *fixed point* of a function f is a value a such that $f(a) = a$. A fixed point a is *attractive* if, for any x sufficiently close to a, the sequence

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))),\ \ldots$$

converges to a. Find the sum of the coordinates of the attractive fixed point of $f(x) = x^3 - 6x^2 + 12x - 6$.

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** NOTA

29. Let s be a root of the equation $z^2 - 3z + 1 = 0$. Compute the sum of all distinct possible values of $s^7 + \frac{1}{s^7}$.

(A) 788 **(B)** 843 **(C)** 896 **(D)** 1009 **(E)** NOTA

30. Denote S to be the set of distinct complex roots to the equation $z^{2020} = 1$. Compute the remainder when

$$\prod_{j \in S} (j^2 + 2j + 2)$$

is divided by 1000.

(A) 72 **(B)** 144 **(C)** 288 **(D)** 576 **(E)** NOTA